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Classical circular strings with pure $SU(2)$ charge

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Abstract. Using Kaluza–Klein-like dimensional reduction techniques we obtain a classical string with a pure $SU(2)$ charge. The string is embedded in the product of Minkowski space and the squashed 3-sphere, which has $SU(2)$ as its isometry group. For certain values of the constants of motion the string can wind around the sphere without contracting to a point, in contrast to what is expected from the topology. This construction finally leads to a non-collapsing circular string in Minkowski space.

1. Introduction

The possible generation of cosmic strings and other topological defects is intimately connected with the phenomenon of spontaneous symmetry breaking [1, 2]. The simplest kind of such a string is a flux tube [3] immersed in a superconducting background. This kind of string may be detected by its gravitational field, which corresponds to a conic singularity in space [4]. It has been suggested [5] that superconducting strings in a symmetric background may also exist. This kind of string may also interact with external electromagnetic field,s thus producing spectacular effects [6]. Strings which interact with non-Abelian gauge fields have also been suggested [7] and their properties in some simple cases have beeen discussed in [8–10].

The method of dealing with a non-Abelian string is via the Kaluza–Klein approach [7], namely letting it evolve in a spacetime which locally looks like a direct product of Minkowski space with a compact manifold X with dimension M. We write the higher-dimensional line-element as

$$ds^2 = G_{MN}\mathrm{d}x^M\mathrm{d}x^N = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + g_{ij}(x^k)\big(\mathrm{d}x^i + A^i_\mu\mathrm{d}x^\mu\big)\big(\mathrm{d}x^j + A^j_\nu\mathrm{d}x^\nu\big) \tag{1}$$

where Greek indices range from 0 to 3, lower case latin indices range from 1 to n and capital latin indices from 0 to $3+n$. A^i_μ is a product of the algebra-valued vector fields with the Killing vectors. The string motion which is described by the 4+n functions $x^M(\tau,\sigma)$ is determined by the Nambu–Goto action

$$S = -\mu\int \mathrm{d}\tau\,\mathrm{d}\sigma\sqrt{-\det G_{AB}} \tag{2}$$

where G_{AB} is the induced metric on the world-sheet. The indices A, B range from 0 to 1.

The non-Abelian properties of the string stem from the choice of the internal manifold X. $X = S^{N-1}$ yields strings with $SO(N)$ charges [8, 9] while the choice $X = CP^{N-1}$ leads

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to $SU(N)/Z_N$ symmetry [10]. The most interesting aspect of these systems is the existence of stable circular solutions due to the self-interaction of the strings and the possibility of non-trivial winding of the string around the internal manifold. It is therefore interesting to extend the analysis to strings with $SU(N)$ charges. The difficulty, which explains why this possibility has not so far been investigated, is in finding the appropriate manifold (and metric tensor) which has $SU(N)$ as its isometry group. A quite simple solution to this question is found in the case $N = 2$ by turning one's attention to the group manifold. Generally the group manifold of $SU(N)$ has a symmetry of $SU(N) \times SU(N)$. It turns out, however, that the group manifold of $SU(2)$ admits a 'squashed' metric which breaks explicitly one $SU(2)$ and leaves us with just the desired $SU(2)$ as a symmetry group.

In this paper we therefore take this path and investigate the dynamical properties of the $SU(2)$ string. After an overview of the squashed $SU(2)$ manifold, given in section 2, an approach for the construction of the effective potential for closed strings is described in section 3. In section 4 this approach is applied to the squashed $SU(2)$ string and its essential properties are discussed. Our conclusions are summarized in section 5.

2. The squashed $SU(2)$

A Lie group G may be endowed with a natural metric tensor known as the Cartan–Killing metric. It is given by

$$g_{ij} = -2\,\mathrm{tr}(L_i L_j) = -2\,\mathrm{tr}(R_i R_j) \tag{3}$$

where L_i and R_i are the left-invariant and right-invariant Maurer–Cartan 1-forms

$$L_i = U^{-1}\partial_i U \qquad R_i = U\partial_i U^{-1}\,. \tag{4}$$

Here ∂_i U are the derivatives of the group elements with respect to its parameters x^i which we think of as coordinates. Since the Maurer–Cartan forms L_i (R_i) are invariant with respect to left (right) multiplication of U by a constant group element, the metric tensor (3) has $G \times G$ as its isometry group. The left-invariant 1-forms may be decomposed into a combination of the (Hermitian) generators of G which we may call λ_i

$$L = L_i \mathrm{d}x^i = l^i \lambda_i \tag{5}$$

(and similarly for R). Thus we obtain $M = \dim(G)$ left-invariant differential 1-forms l^i. Following Duff *et al* [11], we can use them to construct a'squashed' line-element which is only left-invariant, thus having only G as its isometry group. The simplest way of doing that is by introducing M arbitrary positive numbers ρ_i, and constructing the line-element by simple combinations of $(l^i)^2$

$$\mathrm{d}s^2 = \sum_i \rho_i^2 (l^i)^2\,. \tag{6}$$

It is evident that for a general set of ρ_i's the right G-invariance is explicitly broken.

The simplest case which we now discuss is the $SU(2)$ group manifold. First we parametrize $SU(2)$ by Euler angles

$$U = \exp\left(\tfrac{1}{2}\mathrm{i}\,\phi\sigma_z\right)\exp\left(\tfrac{1}{2}\mathrm{i}\,\theta\sigma_y\right)\exp\left(\tfrac{1}{2}\mathrm{i}\,\psi\sigma_z\right) \tag{7}$$

with $0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi \leqslant 2\pi,\ 0 \leqslant \psi \leqslant 4\pi$. Then we compute the left-invariant 1-forms using (5) and find that

$$\begin{aligned} l^1 &= -\sin\psi\,\mathrm{d}\theta + \cos\psi\,\sin\theta\,\mathrm{d}\phi \\ l^2 &= \cos\psi\,\mathrm{d}\theta + \sin\psi\,\sin\theta\,\mathrm{d}\phi \\ l^3 &= \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi. \end{aligned} \tag{8}$$

The line element is given by (6)

$$\begin{aligned} ds^2 &= \rho_1^2(l^1)^2 + \rho_2^2(l^2)^2 + \rho_3^2(l^3)^2 \\ &= \rho_1^2(\sin\psi d\theta - \cos\psi \sin\theta d\phi)^2 + \rho_2^2(\cos\psi d\theta + \sin\psi \sin\theta d\phi)^2 \\ &\quad + \rho_3^2(d\psi + \cos\theta d\phi)^2 \end{aligned} \tag{9}$$

and for non-equal (ρ_1, ρ_2, ρ_3) it is invariant only under left $SU(2)$ generated by the following Killing vectors:

$$\begin{aligned} K^{(1)} &= \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} - \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\psi} \\ K^{(2)} &= \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi} + \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\psi} \\ K^{(3)} &= \frac{\partial}{\partial\phi}. \end{aligned} \tag{10}$$

A more 'symmetric' parametrization of $SU(2)$ that will also be convenient for our purposes is obtained by the redefinitions

$$\phi = \phi_1 + \phi_2 \qquad \psi = \phi_2 - \phi_1 \qquad \rho = \sin\frac{\theta}{2} \tag{11}$$

In this case the group element (7) has the explicit form

$$U = \begin{pmatrix} \sqrt{1-\rho^2}e^{i\phi_2} & \rho e^{i\phi_1} \\ -\rho e^{-i\phi_1} & \sqrt{1-\rho^2}e^{-i\phi_2} \end{pmatrix} \tag{12}$$

where $0 \leqslant \phi_1, \phi_2 \leqslant 2\pi$, $0 \leqslant \rho \leqslant 1$, and both ϕ_1 and ϕ_2 are periodic with period 2π. In section 4 we will use both parametrizations (7) and (12) simultaneously.

3. General analysis

In this section we consider a circular string embedded in the product space $M_4 \times X$, where X is a compact M-dimensional manifold, not necessarily a group manifold. The idea is to write the equations of motion for the string in a Hamiltonian form, which is suitable for further investigation when the exact solution cannot be found analytically. The results of this section will be a slight generalization of section 2 of [12].

The line element is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 + g_{ij}dx^i dx^j \tag{13}$$

with $(i, j) = (1, 2, \ldots, n)$ and the string is, as usual, described by the Nambu–Goto action:

$$S = -\mu \int d\tau\, d\sigma \sqrt{-\det G_{AB}} \tag{14}$$

where G_{AB} is the induced metric on the world-sheet. In a given coordinate system we can write $x^i = (x^\alpha, x^\mu)$ where $\alpha = 1, 2, 3, \ldots, k \leqslant M$ and $\mu = (k+1), \ldots, M$ and

$$g_{ij} = g_{ij}(x^\alpha) = \begin{pmatrix} g_{\alpha\beta} & g_{\alpha\mu} \\ g_{\mu\alpha} & g_{\mu\nu} \end{pmatrix} \tag{15}$$

i.e. we assume that there are $n-k$ cyclic coordinates in the internal space. The ansatz describing a circular string in Minkowski space is taken to be

$$t=\tau \qquad r=r(\tau) \qquad z=0 \qquad \theta=\sigma \qquad x^{\alpha}=x^{\alpha}(\tau) \qquad \partial x^{\mu}/\partial\sigma=n^{\mu} \tag{16}$$

where n^{μ} are constants ('winding numbers'). It is now straightforward to derive the equations of motion for the dynamical variables (cf [12])

$$\frac{\dot{r}^2}{\omega^2}+r^2+\frac{L^2}{r^2}=c_1^2 \tag{17}$$

$$\dot{x}^{\mu}=\omega\left[g^{\mu\nu}\Omega_{\nu}+\frac{Ln^{\mu}}{r^2}\right]-g^{\mu\nu}g_{\nu\alpha}\dot{x}^{\alpha} \tag{18}$$

where ω, L, c_1 and Ω_{μ} are integration constants, and Ω_{μ} the constant 'charges'. Finally the x^{α}'s are determined by

$$h_{\alpha\beta}\ddot{x}^{\beta}+\partial_{\gamma}h_{\alpha\beta}\dot{x}^{\beta}\dot{x}^{\gamma}-\tfrac{1}{2}\partial_{\alpha}h_{\gamma\beta}\dot{x}^{\beta}\dot{x}^{\gamma}+\tfrac{1}{2}\omega^2\partial_{\alpha}[g_{\mu\nu}(n^{\mu}n^{\nu}+\Omega^{\mu}\Omega^{\nu})]$$
$$-\omega\dot{x}^{\gamma}[\partial_{\alpha}(\Omega^{\mu}g_{\mu\gamma})-\partial_{\gamma}(\Omega^{\mu}g_{\mu\alpha})]=0\,. \tag{19}$$

Here $\Omega^{\mu}\equiv g^{\mu\nu}\Omega_{\nu}$ and $h_{\alpha\beta}\equiv g_{\alpha\beta}-g^{\mu\nu}g_{\nu\beta}g_{\mu\alpha}$. It is easy to show that the latter can be derived as Hamilton equations from the Hamiltonian

$$H(x^{\alpha},p_{\alpha})=(1/2\omega^2)h^{\alpha\beta}(p_{\alpha}-\omega\Omega_{\mu}A^{\mu}_{\alpha})(p_{\beta}-\omega\Omega_{\nu}A^{\nu}_{\beta})+V_{\text{eff}}(x^{\alpha})\equiv c_2^2/2 \tag{20}$$

where $A^{\mu}_{\alpha}\equiv g^{\mu\nu}g_{\nu\alpha}$, $V_{\text{eff}}\equiv(g_{\mu\nu}/2)[n^{\mu}n^{\nu}+\Omega^{\mu}\Omega^{\nu}]$ and c_2^2 is the constant 'energy'. Finally the various integration constants are constrained by $c_1^2+c_2^2=1/\omega^2$ and $L=n^{\mu}\Omega_{\mu}$. Note that the terms mixing cyclic and non-cyclic coordinates in the line element show up in the Hamiltonian as a μ-vector of Abelian potentials. This is of course not at all surprising as can be seen as follows. First write the metric (14) in the equivalent way

$$g_{ij}=\begin{pmatrix} h_{\alpha\beta}+g_{\mu\nu}A^{\mu}_{\alpha}A^{\nu}_{b}eta & g_{\mu\rho}A^{\rho}_{\alpha} \\ g_{\mu\rho}A^{\rho}_{\alpha} & g_{\mu\nu}\end{pmatrix}. \tag{21}$$

Then observe that, since A^{μ}_{α}, $g_{\mu\nu}$ and $h_{\alpha\beta}$ depend on x^{α} only, this is exactly the metric used in the 'ordinary dimensional reduction' scheme of Schwarz and Scherk [13] leading to an $[U(1)]^{n-k}$ invariance. It should however be stressed that in our case all this takes place in the internal space so from the four-dimensional point of view A^{μ}_{α} has absolutely nothing to do with an electromagnetic potential. Our construction can therefore be characterized as an 'ordinary dimensional reduction' taking place inside a 'generalized dimensional reduction' or just as 'Kaluza–Klein in Kaluza–Klein'.

In section 4 we will use this general formalism to analyse a string with an internal $SU(2)$-charge.

4. Analysis of the minima

For a circular string embedded in the product of Minkowski space and the squashed 3-sphere (squashed $SU(2)$) we make the following ansatz in the internal space, using the parametrization (11), (12):

$$\rho=\rho(\tau) \qquad \phi_1=B_1(\tau)+n\sigma \qquad \phi_2=B_2(\tau)+n\sigma \tag{22}$$

where n is the winding number. Note that we need the same winding number in the two 'angular' coordinates since only the combination $\phi=\phi_1+\phi_2$ is cyclic; different winding numbers will lead to explicit σ-dependence of the equations of motion which will then correspond to another family of solutions. We will however only consider solutions of the general form described in section 3. For simplicity we will furthermore only consider different 'squashing-parameters' in the form (ρ_2 scaled to 1)

$$(\rho_1, \rho_2, \rho_3) \equiv (\lambda, 1, \mu) \qquad \mu^2 > \lambda^2 > 1\,. \tag{23}$$

Other possibilities will eventually lead to similar conclusions.

For $\rho = 1$ and $\rho = 0$ the line element (9) degenerates into $ds^2 = 4\mu^2 d\phi_1^2$ and $ds^2 = 4\mu^2 d\phi_2^2$, respectively, which in both cases means that the string is collapsed in one of the angular coordinates. In that case the winding number in (21) is ill-defined so it is important that the string is kept away from these 'poles'. Obviously the topology itself does not prevent the string from collapsing since squashing the 3-sphere does not change the topology (trivial fundamental group), but it will turn out that it is actually possible to construct solutions where the dynamics of the string prevents it from collapsing, as we will see in a moment.

To use the general formalism of section 3 on the squashed sphere of section 2 we make the identifications

$$x^i = (\rho, \psi, \phi) \qquad x^\alpha = (\rho, \psi) \qquad x^\mu = \phi \qquad n^\mu = 2n \tag{24}$$

i.e. ϕ is the cyclic coordinate and (ρ, ψ) are the non-cyclic. The various elements $(g_{\mu\nu}, g_{\alpha\beta}, g_{\mu\alpha})$ of the metric of the squashed 3-sphere can then be read off from the line element (9) of section 2. Furthermore we also obtain the equations of motion directly. The string radius in Minkowski space is as always determined by equation (17) with $L = 2n\Omega$, and the cyclic coordinate $B \equiv B_1 + B_2$ fulfils equation (18)

$$\dot{B} = \omega\left[\frac{\Omega}{g_{\phi\phi}} + \frac{2nL}{r^2}\right] - \frac{1}{g_{\phi\phi}}[g_{\phi\rho}\dot{\rho} + g_{\phi\psi}\dot{\psi}]. \tag{25}$$

This equation can of course only be solved provided that we know ρ and ψ, which are finally determined by a Hamiltonian of the form (20). The effective potential is here given explicitly by

$$V_{\text{eff}} = \frac{1}{2}\left(4n^2 g_{\phi\phi} + \frac{\Omega^2}{g_{\phi\phi}}\right) \tag{26}$$

where from (9), (11)

$$g_{\phi\phi} = \mu^2 + 4\rho^2(1-\rho^2)[\sin^2\psi + \lambda^2\cos^2\psi - \mu^2]\,. \tag{27}$$

We now want to show that this effective potential admits stable solutions away from the poles ($\rho = 0$, $\rho = 1$) so that the winding number, which together with the charge Ω guarantees the non-collapsing nature in Minkowski space via the angular momentum term in (17), is well defined. Since $\psi = 0$ is to be identified with $\psi = 4\pi$ the potential is defined on a cylinder with boundaries given by $\rho = 0$ and $\rho = 1$. The potential is furthermore periodic around the cylinder with a period of π. The critical points inside the surface of the cylinder are given by

(i) $\quad \rho^2 = \frac{1}{2} \qquad \cos\psi = 0 \qquad V_{\text{eff}} = (4n^2 + \Omega^2)/2$

(ii) $\quad \rho^2 = \frac{1}{2} \qquad \sin\psi = 0 \qquad V_{\text{eff}} = (4n^2\lambda^2 + \Omega^2/\lambda^2)/2$

(iii) $\quad g_{\phi\phi} = \left|\frac{\Omega}{2n}\right| \qquad V_{\text{eff}} = |2n\Omega|$

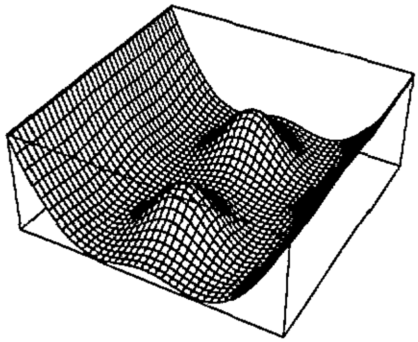

Figure 1. The effective potential for $\nu^2 \in]\mu^2, \lambda^2[$. We show the potential for $\psi \in [0, 2\pi]$. The potential is periodic with a period π and $\psi = 0$ has to be identified with $\psi = 4\pi$. The parameter values are $\mu^2 = 5$, $\nu^2 = 2$, $\lambda^2 = \frac{3}{2}$ and the minima are two closed curves winding around the cylinder.

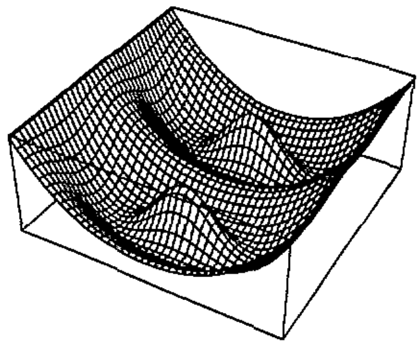

Figure 2. The effective potential for $\nu^2 \in]\lambda^2, 1[$. We show the potential for $\psi \in [0, 2\pi]$. The potential is periodic with a period π and $\psi = 0$ has to be identified with $\psi = 4\pi$. The parameter values are $\mu^2 = 5$, $\lambda^2 = 3$, $\nu^2 = 2$ and the minima are four closed curves at the surface of the cylinder.

whereas at the boundaries ($\rho = 0, \rho = 1$) we have $V_{\rm eff} = (4n^2\mu^2 + \Omega^2/\mu^2)/2$. It follows that if there are solutions in the third case then this critical 'curve' represents the absolute minimum, since it always has the lowest potential.

Now if we can fine-tune the various constants of motion such that the boundaries represent unstable configurations then we can always construct stable solutions where the string is kept away from the poles. We therefore consider expansions of the potential near the boundaries

$$V_{\rm eff}(\rho, \psi) - V_{\rm eff}(0, \psi) \approx \rho^2(8n^2 - 2\Omega^2/\mu^4)F(\psi) + \rho^4[8\Omega^2F^2(\psi)/\mu^6 - (8n^2 - 2\Omega^2/\mu^4)F(\psi)] + \cdots \tag{28}$$

$$V_{\rm eff}(\rho, \psi) - V_{\rm eff}(1, \psi) \approx 2(1-\rho)(8n^2 - 2\Omega^2/\mu^4)F(\psi) + (\rho - 1)^2[32\Omega^2F^2(\psi)/\mu^6 - 5(8n^2 - 2\Omega^2/\mu^4)F(\psi)] + \cdots \tag{29}$$

where

$$F(\psi) \equiv \sin^2\psi + \lambda^2\cos^2\psi - \mu^2 < 0\,. \tag{30}$$

Using (23) it is then clear that the poles $\rho = 0$ and $\rho = 1$ are unstable provided

$$\mu^2 > \left|\frac{\Omega}{2n}\right| \equiv \nu^2\,. \tag{31}$$

If this inequality is fulfilled we can now construct stable configurations with well-defined winding numbers, just by taking the 'energy' c_2^2 sufficiently low. Let us finally analyse the structure of minima in a little more detail. Suppose for instance that the squashing parameters λ^2 and μ^2 are kept fixed ($\mu^2 > \lambda^2 > 1$). Then decrease ν^2 continuously downwards from μ^2 to zero, corresponding to (say) decreasing the $SU(2)$-charge Ω. For $\nu^2 \in]\mu^2, \lambda^2[$ the minima are represented by two continuous curves winding around the cylinder (figure 1). At $\nu^2 = \lambda^2$ the two curves break into four closed curves on the cylinder surrounding the four unstable points ($\rho^2 = \frac{1}{2}, \cos\psi = 0$). For $\nu^2 \in]\lambda^2, 1[$ these four curves

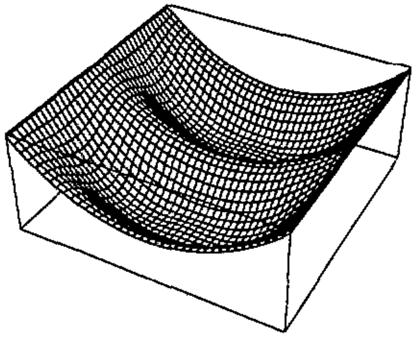

Figure 3. The effective potential for $\nu^2 \leqslant 1$. We show the potential for $\psi \in [0, 2\pi]$. The potential is periodic with a period π and $\psi = 0$ has to be identified with $\psi = 4\pi$. The parameter values are $\mu^2 = 5$, $\lambda^2 = 3$, $\nu^2 = 2/3$ and the minima are four discrete points.

shrink for lower and lower ν^2 (figure 2), and finally for $\nu^2 \leqslant 1$ the minima are just the four discrete stable points ($\rho^2 = \frac{1}{2}, \cos\psi = 0$) (figure 3).

It should be stressed that the squashing parameters λ and μ are not directly measurable as physical quantities in this approach, since they are parameters in the internal manifold. The physical aspects are related to equation (17) determining the dynamics in Minkowski space, so for an actual $SU(2)$ string one might think of measuring for instance the average radius etc. What we have shown in this section, however, is that, if such a stable circular $SU(2)$-charged string is found, it may perhaps be described by a mathematical model of the Kaluza–Klein type with a suitably squashed 3-sphere as internal manifold.

5. Conclusion

The main result of this paper is the existence of stable circular strings with pure $SU(2)$ charges. There is, however, an important difference with respect to the systems studied previously [8–10]. In the case of the $SO(N)$ and $SU(N)/Z_N$ strings the motion in the internal space is stabilized by an infinite repulsive force which protects the string from collapsing to a point. In the $SU(2)$ case the analogous repulsive force is finite. This fact does not exclude stable solutions but these solutions are evidently more 'fragile' than the others. The structure of the stability regions (the potential minima) which is determined by the $SU(2)$ charge and winding number as well as by the 'squashing parameters' is, as we saw in the previous section, quite involved. The connection between its degeneracy and the $SU(2)$ representations is unclear and deserves further investigation.

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